Vector-valued modular forms on finite upper half planes*

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Abstract. Finite upper half planes are finite field analogs of the Poincaré upper half plane. Vector-valued modular forms on finite upper half planes are introduced, and then equivariant functions on these planes are defined. The existence of these functions is an application of vector-valued modular forms.

Keywords: vector-valued modular form · equivariant function · finite upper half plane

1 Introduction

Let $SL(2, \mathbb{Z})$ be the classical modular group. This group acts on the Poincaré upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ by the linear fractional transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Let $\rho : SL(2, \mathbb{Z}) \to GL(n, \mathbb{C})$ be an n-dimensional complex representation. A holomorphic map $F : \mathfrak{H} \to \mathbb{C}^n$ is called a vector-valued modular form of weight w (w any real number) and multiplier ρ if for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, we have

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^w \rho(\begin{pmatrix} a & b \\ c & d \end{pmatrix})F(z),$$

and if a cuspidal condition holds [2, 11, 12]. The classical vector-valued modular forms have been investigated as a generalization of scalar-valued modular forms. As pointed out by Selberg [16], these modular forms can be used in the study of modular forms for finite index subgroups of $SL(2, \mathbb{Z})$. The Jacobi forms developed by Eichler and Zagier [6] are related to these modular forms. In physics, they appear as the characters in rational conformal field theory [5, 7].

A meromorphic function h on $\mathfrak{H}$ is called an *equivariant function* for $SL(2, \mathbb{Z})$ if it satisfies the condition

$$h\left(\frac{az + b}{cz + d}\right) = \frac{ah(z) + b}{ch(z) + d}$$

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for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathfrak{H}$. Such a function is related to modular forms [13–15].

In the mid-1980s, A. Terras introduced a finite upper half plane H_q that is defined over a finite field $\mathbb{F}_q$ as an analog of the Poincaré upper half plane $\mathfrak{H}$. Specifically, she and her coworkers investigated special functions on H_q in [1, 3, 18, 19]. In [9], modular forms of a new type were studied on H_q. In the present paper, modular forms of other types are considered on H_q. Generalized and subsequently vector-valued modular forms are introduced. In particular, the definition of vector-valued modular forms is new. Moreover, when q is a prime number p, for a complex representation $\rho : GL(2, \mathbb{F}_p) \to GL(2, \mathbb{C})$, equivariant functions on H_p are defined. The existence of these functions is an application of vector-valued modular forms.

Notation. For a field F, let $F^\times = F \setminus \{0\}$.

2 Generalized modular forms

In this section, the generalized modular forms on finite upper half planes are introduced. For the classical modular forms, the reader is referred to [10].

2.1 Generalized modular forms

Let q be a power of an odd prime number p, and let $\mathbb{F}_q$ be the finite field with q elements. Let a non-square element $\delta \in \mathbb{F}_q$ be fixed, and let

$$H_q = \{z = x + y\sqrt{\delta} \mid x, y \in \mathbb{F}_q, y \neq 0\},$$

which is called a *finite upper half plane*. This plane is a finite field analog of the Poincaré upper half plane $\mathfrak{H}$. It should be noted that $\sqrt{\delta}$ plays the role of $i = \sqrt{-1}$ in $\mathfrak{H}$ and that H_q is a subset of $\mathbb{F}_q(\sqrt{\delta})$, which is analogous to the fact that $\mathfrak{H}$ is a subset of the field of complex numbers $\mathbb{C} = \mathbb{R}(i)$. For $z = x + y\sqrt{\delta} \in H_q$, let

$$x = \mathrm{Re}(z),\ y = \mathrm{Im}(z),\ \overline{z} = x - y\sqrt{\delta},\ N(z) = z\overline{z} = x^2 - \delta y^2,\ \mathrm{Tr}(z) = z + \overline{z} = 2x.$$

Moreover, let $G_q = GL(2, \mathbb{F}_q)$ be the general linear group over $\mathbb{F}_q$. This group acts on H_q by the following linear fractional transformation: for $z \in H_q$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_q$, let

$$\gamma z = \frac{az + b}{cz + d}.$$

The fixed subgroup of $\sqrt{\delta}$ in G_q is

$$K_q = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \mid a, b \in \mathbb{F}_q, a^2 - \delta b^2 \neq 0 \right\},$$

which is an analog of the orthogonal group $O(2)$. It is known that the action of G_q on H_q is transitive. Hence, H_q is expressed as $H_q = G_q/K_q$.

Let Γ be a subgroup of G_q. The map $m : \Gamma \times H_q \to \mathbb{C}^\times$ is called a *multiplier system* for Γ if

$$m(\gamma\gamma', z) = m(\gamma, \gamma' z)m(\gamma', z)$$

holds for all $\gamma, \gamma' \in \Gamma$ and $z \in H_q$. In the classical case, the definition of a multiplier system is wider, as in [4, 8]. However, in this paper, the definition in [1, 9, 18, 19] was used. Let $\mu : G_q \to \mathbb{C}^\times$ be a multiplicative character. For these Γ, m, and μ, a $\mathbb{C}$-valued function $f : H_q \to \mathbb{C}$ is called a *generalized modular form* for Γ, m, and μ if for any $\gamma \in \Gamma$, it holds that

$$f(\gamma z) = m(\gamma, z)\mu(\gamma)f(z).$$

The space of generalized modular forms of this type is denoted by $M(\Gamma, m, \mu)$. When μ is a trivial character, a generalized modular form, which is called a *modular form*, was studied in [1, 9, 18, 19].

2.2 Special Cases

Herein, the generalized modular forms are discussed when Γ is the unipotent subgroup of G_q or K_q.

2.2.1 Case $\boldsymbol{\Gamma = N_q}$

Let $q = p$ be an odd prime number, and let

$$N_p = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{F}_p \right\}.$$

Let $\chi : \mathbb{F}_p^\times \to \mathbb{C}^\times$ be a multiplicative character, and let $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ be an additive character. Using these, a function on H_p is defined by

$$f(z; \chi, \psi) = \sum_{u \in \mathbb{F}_p} \chi\left(\mathrm{Im}\left(\frac{-1}{z + u}\right)\right)\psi(u). \tag{1}$$

This function, which is an analog of the classical K-Bessel function, was first defined in [3].

Theorem 1. (i) *For* $\gamma_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N_p$, *we have*

$$f(\gamma_u z; \chi, \psi) = \mu(\gamma_u)f(z; \chi, \psi),$$

where $\mu : N_p \to \mathbb{C}^\times$ *is the character defined by* $\mu(\gamma_u) = \psi(-u)$. *That is,* $f(z; \chi, \psi) \in M(N_p, 1, \mu)$.

(ii) *If* χ *and* ψ *are non-trivial, then* $f(z; \chi, \psi)$ *is non-zero.*

Proof. See [3, Lemma 3]. □

2.2.2 Case $\Gamma = K_q$

Let $\pi : \mathbb{F}_q(\sqrt{\delta})^\times \to \mathbb{C}^\times$ be a multiplicative character. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_q$, let $J_\pi(\gamma, z) = \pi(cz + d)$. Then, for $\gamma, \gamma' \in K_q$, it holds that

$$J_\pi(\gamma\gamma', z) = J_\pi(\gamma, \gamma' z) J_\pi(\gamma', z). \tag{2}$$

Hence, $J_\pi : K_q \times H_q \to \mathbb{C}^\times$ is a multiplier system for K_q. A map $m : K_q \times H_q \to \mathbb{C}^\times$ is defined by

$$m(k, z) = \frac{J_\pi(k, z)}{J_\pi(k, \sqrt{\delta})}. \tag{3}$$

It is easy to prove that m is a multiplier system for K_q. For a multiplicative character $\mu : K_q \to \mathbb{C}^\times$, let

$$E(z; \pi, \mu) = \frac{1}{|K_q|} \sum_{k \in K_q} \frac{J_\pi(k, \sqrt{\delta})}{J_\pi(k, z)} \cdot \mu(k)^{-1}, \tag{4}$$

where $|K_q|$ is the number of elements of K_q. $E(z; \pi, \mu)$ is called the *Eisenstein sum* for K_q, π, and μ. This is a finite field analog of the Eisenstein series on the Poincaré upper half plane.

Theorem 2. *$E(z; \pi, \mu) \in M(K_q, m, \mu)$.*

Proof. By (2), for $k' \in K_q$, it holds that

$$\begin{aligned} |K_q| E(k' z; \pi, \mu) &= \sum_{k \in K_q} \frac{J_\pi(kk', \sqrt{\delta})}{J_\pi(k', \sqrt{\delta})} \cdot \frac{J_\pi(k', z)}{J_\pi(kk', z)} \cdot \mu(k)^{-1} \\ &= m(k', z)\mu(k') \sum_{k \in K_q} \frac{J_\pi(kk', \sqrt{\delta})}{J_\pi(kk', z)} \cdot \mu(kk')^{-1}, \end{aligned}$$

which yields the result. □

In general, it is difficult to determine the dimension of $M(\Gamma, m, \mu)$ over $\mathbb{C}$. Using Eisenstein sums, an easy example may be provided.

Example 1. Let $q = p = 3$, $\delta = -1$, and $i = \sqrt{-1}$. Then, $1 + \sqrt{\delta}$ is a generator of $\mathbb{F}_3(\sqrt{\delta})^\times$. The multiplicative character $\pi : \mathbb{F}_q(\sqrt{\delta})^\times \to \mathbb{C}^\times$ is defined by $\pi(1 + \sqrt{\delta}) = \exp(2\pi i/8)$. The group K_3 is a cyclic group generated by $g = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The multiplicative character $\mu : K_3 \to \mathbb{C}^\times$ is defined by $\mu(g) = \exp(2\pi i/4)^{-1} = -i$. H_3 can be decomposed into a union of K_3-orbits as follows:

$$H_3 = \{\sqrt{\delta}\} \cup \{-\sqrt{\delta}\} \cup \{\pm 1 \pm \sqrt{\delta}\}.$$

Using π, a multiplier system $m_4 : K_3 \times H_3 \to \mathbb{C}^\times$ is defined by

$$m_4(g^j, z) = \frac{J_{\pi^4}(g^j, z)}{J_{\pi^4}(g^j, \sqrt{\delta})} \qquad (z \in H_3, j = 1, \ldots, 8).$$

Then, $m_4(g^j, \sqrt{\delta}) = m_4(g^j, -\sqrt{\delta}) = 1$ for $j = 1, \ldots, 8$. Each modular form $f \in M(K_3, m_4, \mu)$ is determined by the values $f(\sqrt{\delta})$, $f(-\sqrt{\delta})$, and $f(1+\sqrt{\delta})$. For $f \in M(K_3, m_4, \mu)$, we have

$$f(\sqrt{\delta}) = m(g, \sqrt{\delta})\mu(g) f(\sqrt{\delta}) = -i f(\sqrt{\delta}),$$

which implies that $f(\sqrt{\delta}) = 0$. Similarly, it follows that $f(-\sqrt{\delta}) = 0$. By definition, we have

$$E(1+\sqrt{\delta}; \pi^4, \mu) = \frac{1}{8}\sum_{j=1}^{8} \frac{J_{\pi^4}(g^j, \sqrt{\delta})}{J_{\pi^4}(g^j, 1+\sqrt{\delta})} \cdot \mu(g)^{-1} = \frac{1+i}{2}.$$

Consequently, $\dim_{\mathbb{C}} M(K_3, m_4, \mu) = 1$.

3 Vector-valued modular forms

In this section, to extend the generalized modular forms defined in Section 2, vector-valued modular forms on finite upper half planes are introduced. For the classical vector-valued modular forms, the reader is referred to [2, 11, 12].

3.1 Vector-valued modular forms

Let $\mathcal{F}_q$ be the set of all $\mathbb{C}$-valued functions on H_q, Γ be a subgroup of G_q, and $\rho : \Gamma \to GL_n(\mathbb{C})$ be an n-dimensional complex representation. A *vector-valued modular form* for Γ, the multiplier system $m : \Gamma \times H_q \to \mathbb{C}^{\times}$, and ρ is an element $F(z) = (f_1(z), \ldots, f_n(z))^t \in \mathcal{F}_q^n$ satisfying

$$F(\gamma z) = m(\gamma, z)\rho(\gamma)F(z)$$

for $\gamma \in \Gamma$ and $z \in H_q$. The space of all vector-valued modular forms of this type is denoted by $M(\Gamma, m, \rho)$.

The following is a basic result.

Theorem 3. *Let Γ be a subgroup of G_q.*

(i) *For two complex representations ρ_1 and ρ_2 of Γ, there exists a linear isomorphism*

$$M(\Gamma, m, \rho_1) \oplus M(\Gamma, m, \rho_2) \cong M(\Gamma, m, \rho_1 \oplus \rho_2).$$

(ii) *Let $\rho : \Gamma \to GL(n, \mathbb{C})$ be an n-dimensional complex representation. For $F_i \in M(\Gamma, m_i, \rho)$ $(i = 1, \ldots, n)$, let $F = (F_1, \ldots, F_n)$. Then,* $\det F \in M(\Gamma, m_1 \cdots m_n, \det \rho)$.

(iii) *Let Γ be an abelian subgroup of G_q, and let $\rho : \Gamma \to GL(n, \mathbb{C})$ be a complex representation. For any vector-valued modular form $F(z) \in M(\Gamma, m, \rho)$, there exists $U \in GL(n, \mathbb{C})$ such that $UF(z)$ can be written as a direct sum of some generalized modular forms.*

Proof. (i) is immediate from the definition of vector-valued modular forms.

(ii) For $\gamma \in \Gamma$,

$$F(\gamma z) = (m_1(\gamma, z)\rho(\gamma)F_1(z), \ldots, m_n(\gamma, z)\rho(\gamma)F_n(z))$$
$$= \rho(\gamma)F(z) \begin{pmatrix} m_1(\gamma, z) & & O \\ & \ddots & \\ O & & m_n(\gamma, z) \end{pmatrix},$$

which yields the result.

(iii) By assumption, there exist $U \in GL(n, \mathbb{C})$ and 1-dimensional representations $\mu_1, \ldots, \mu_n : \Gamma \to \mathbb{C}^\times$ such that for all $\gamma \in \Gamma$,

$$\rho(\gamma) = U^{-1} \begin{pmatrix} \mu_1(\gamma) & & O \\ & \ddots & \\ O & & \mu_n(\gamma) \end{pmatrix} U.$$

Let $UF(z) = (f_1(z), \ldots, f_n(z))^t$. Then, it holds that for all $\gamma \in \Gamma$,

$$\begin{pmatrix} f_1(\gamma z) \\ \vdots \\ f_n(\gamma z) \end{pmatrix} = \begin{pmatrix} m(\gamma, z)\mu_1(\gamma)f_1(z) \\ \vdots \\ m(\gamma, z)\mu_n(\gamma)f_n(z) \end{pmatrix}.$$

□

For a subgroup Γ of G_q, let $m : \Gamma \times H_q \to \mathbb{C}^\times$ be a multiplier system. For $f \in \mathcal{F}_q$ and $\gamma \in \Gamma$, the map $\mathcal{F}_q \times \Gamma \to \mathcal{F}_q$, $(f, \gamma) \mapsto f(\gamma z)m(\gamma, z)^{-1}$ defines a right action of Γ on $\mathcal{F}_q$. Using this action, we have the following result, which is an analog of a result proved in [11].

Theorem 4. *Let $M \subset \mathcal{F}_q$ be a finite dimensional Γ-module with generators $f_1, \ldots, f_n$. Then, there exists an n-dimensional complex representation $\rho : \Gamma \to GL(n, \mathbb{C})$ such that $F(z) = (f_1(z), \ldots, f_n(z))^t \in M(\Gamma, m, \rho)$.*

Proof. Changing the order of $f_1, \ldots, f_n$, it may be assumed that $\{f_1, \ldots, f_d\}$ is a basis of M, and that $f_{d+1}, \ldots, f_n$ are written as linear combinations of $f_1, \ldots, f_d$. Let $G = (f_1, \ldots, f_d)^t$. For any $\gamma \in \Gamma$, there exists a unique element $a_{jk}(\gamma) \in \mathbb{C}$ such that

$$f_j(\gamma z)m(\gamma, z)^{-1} = \sum_{k=1}^{d} a_{jk}(\gamma)f_k(z) \quad (j = 1, \ldots, d).$$

Letting $a(\gamma) = (a_{jk}(\gamma))$, a d-dimensional complex representation $a : \Gamma \to GL(d, \mathbb{C})$ is obtained. Using this, we have $G(\gamma z)m(\gamma, z)^{-1} = a(\gamma)G(z)$ for $\gamma \in \Gamma$ and $z \in H_q$.

Let $e = n - d$. By assumption, there exists a matrix $Q \in \mathrm{Mat}_{e\times d}(\mathbb{C})$ such that $(f_{d+1}, \ldots, f_n)^t = QG$. Let $P = \begin{pmatrix} I_d \\ Q \end{pmatrix}$. Then, there exists a matrix $R \in$

$\mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $RP = \begin{pmatrix} I_d \\ O \end{pmatrix}$. An n-dimensional complex representation $\rho : \Gamma \to GL(n, \mathbb{C})$ is defined by

$$\rho(\gamma) = R^{-1} \begin{pmatrix} a(\gamma) & O \\ O & I_e \end{pmatrix} R.$$

As

$$F = \begin{pmatrix} G \\ f_{d+1} \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} I_d \\ Q \end{pmatrix} G = PG,$$

it follows that for $\gamma \in \Gamma$,

$$\begin{aligned} F(\gamma z)m(\gamma, z)^{-1} &= PG(\gamma z)m(\gamma, z)^{-1} = Pa(\gamma)G = R^{-1} \begin{pmatrix} a(\gamma) \\ O \end{pmatrix} G \\ &= R^{-1} \begin{pmatrix} a(\gamma) & O \\ O & I_e \end{pmatrix} \begin{pmatrix} I_d \\ O \end{pmatrix} G = \rho(\gamma)PG = \rho(\gamma)F. \end{aligned}$$

□

Example 2. A vector-valued Maass Eisenstein series on H_q is introduced to generalize the scalar-valued Maass Eisenstein series defined in [17]. Let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a multiplicative character. Let Γ be a subgroup of G_q, an n-dimensional complex representation $\rho : \Gamma \to GL(n, \mathbb{C})$ is considered, and a vector $\mathbf{a} \in \mathbb{C}^n$ is chosen. The *vector-valued Maass Eisenstein sum* for Γ, χ, ρ, and $\mathbf{a}$ is defined by

$$E_\Gamma(z; \chi, \rho, \mathbf{a}) = \sum_{\gamma \in \Gamma} \chi\left(\mathrm{Im}(\gamma z)\right) \rho(\gamma)^{-1} \mathbf{a}^t.$$

It is easily seen that for $\gamma' \in \Gamma$,

$$E_\Gamma(\gamma' z; \chi, \rho, \mathbf{a}) = \rho(\gamma') E_\Gamma(z; \chi, \rho, \mathbf{a}),$$

which implies that $E_\Gamma(z; \chi, \rho, \mathbf{a}) \in M(\Gamma, 1, \rho)$.

Let $n = 2$. When $F(z) := \sum_{\gamma \in \Gamma} \chi\left(\mathrm{Im}(\gamma z)\right) \rho(\gamma)^{-1}$ is not zero, there exists $E_\Gamma(z; \chi, \rho, \mathbf{a})$ whose lowest component is not zero. Indeed, if the $(2, 1)$-entry of $F(z)$ is not zero, then $\mathbf{a}$ may be chosen to be $(1, 0)$. If the $(2, 2)$-entry of $F(z)$ is not zero, then $\mathbf{a}$ may be chosen to be $(0, 1)$. If the $(1, 1)$-entry of $F(z)$ is not zero, then by replacing ρ with $\rho' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$, $\mathbf{a}$ may be chosen to be $(0, 1)$. If the $(1, 2)$-entry of $F(z)$ is not zero, then by replacing ρ with $\rho' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1}$, $\mathbf{a}$ may be chosen to be $(1, 0)$.

3.2 Special cases

Herein, vector-valued modular forms are discussed when Γ is N_q or K_q.

3.2.1 Case $\Gamma = N_q$ To construct a vector-valued modular forms, a generalization of the Gauss sum is introduced. Let $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a multiplicative character, and let $\psi : \mathbb{F}_q \to GL(n, \mathbb{C})$ be a group homomorphism. Using these, the *matrix-valued Gauss sum* $G_q(\chi, \psi)$ is defined by

$$G_q(\chi, \psi) = \sum_{c \in \mathbb{F}_q^\times} \chi(c)\psi(c) \qquad (\in \mathrm{Mat}_{n\times n}(\mathbb{C}))\,.$$

The definition of this Gauss sum may not be new; however, the author is unfamiliar with the related references. The following proposition is easy to prove.

Proposition 1. *Let $\chi_0 : \mathbb{F}_q^\times \to \mathbb{C}^\times$ be a trivial multiplicative character, and let $\psi_0 : \mathbb{F}_q \to GL(n, \mathbb{C})$ be a trivial group homomorphism. Then,*

(i)

$$G_q(\chi, \psi_n) = \begin{cases} (q-1)I_n & if\ \chi = \chi_0 \text{ and } \psi_n = \psi_0, \\ -I_n & if\ \chi = \chi_0 \text{ and } \psi_n \neq \psi_0, \\ O_n & if\ \chi \neq \chi_0 \text{ and } \psi_n = \psi_0. \end{cases}$$

(ii) *If $\chi \neq \chi_0$ and $\psi_n \neq \psi_0$, then $\overline{G_q(\chi, \psi)}G_q(\chi, \psi) = qI_n$, where $\overline{G_q(\chi, \psi)}$ is the complex conjugate of $G_q(\chi, \psi)$.*

Henceforth, let $q = p$ and $n = 2$. For a multiplicative character $\chi : \mathbb{F}_p^\times \to \mathbb{C}^\times$, a group homomorphism $\psi : \mathbb{F}_p \to GL(2, \mathbb{C})$, and a vector $\mathbf{a} \in \mathbb{C}^2$, a function $F(z; \chi, \psi, \mathbf{a})$ on H_q is defined by

$$F(z; \chi, \psi, \mathbf{a}) = \sum_{u \in \mathbb{F}_p} \chi\left(\mathrm{Im}\left(\frac{-1}{z+u}\right)\right)\psi(u)\mathbf{a}^t,$$

which is a generalization of (1).

Theorem 5. (i) *For $\gamma_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N_p$, we have*

$$F(\gamma_u z; \chi, \psi, \mathbf{a}) = \rho(\gamma_u)F(z; \chi, \psi, \mathbf{a}),$$

where $\rho : N_p \to GL(2, \mathbb{C})$ is a 2-dimensional complex representation defined by $\rho(\gamma_u) = \psi(-u)$. That is, $F(z; \chi, \psi, \mathbf{a}) \in M(N_p, 1, \rho)$.

(ii) *If χ and ψ are non-trivial, then there exists a vector $\mathbf{a} \in \mathbb{C}^2$ such that $F(z; \chi, \psi, \mathbf{a})$ is non-zero.*

Proof. (i) When $z = x + y\sqrt{\delta}$, it holds that

$$\mathrm{Im}\left(\frac{-1}{z+u}\right) = \frac{y}{(x+u)^2 - \delta y^2},$$

which yields

$$F(z;\chi,\psi,\mathbf{a}) = \sum_{u\in\mathbb{F}_p} \chi(y)\overline{\chi((x+u)^2-\delta y^2)}\psi(u)\mathbf{a}^t$$
$$= \chi(y)\psi(-x)\sum_{v\in\mathbb{F}_p} \overline{\chi(v^2-\delta y^2)}\psi(v)\mathbf{a}^t.$$

Hence, it holds that

$$F(z+u;\chi,\psi,\mathbf{a}) = \psi(-u)F(z;\chi,\psi,\mathbf{a}).$$

From this, the result follows.

(ii) Using (i), we obtain

$$\begin{aligned}\sum_{y\in\mathbb{F}_p^\times} \overline{\chi}(y)\psi(x)F(z;\chi,\psi,\mathbf{a}) &= \sum_{\substack{y\in\mathbb{F}_p^\times \\ v\in\mathbb{F}_p}} \overline{\chi(v^2-\delta y^2)}\psi(v)\mathbf{a}^t \\ &= \sum_{\substack{w\in\mathbb{F}_{p^2} \\ \mathrm{Im}(w)\neq 0}} \overline{\chi}(N(w))\psi\left(\frac{1}{2}\mathrm{Tr}(w)\right)\mathbf{a}^t \\ &= \sum_{w\in\mathbb{F}_{p^2}} \overline{\chi}(N(w))\psi\left(\frac{1}{2}\mathrm{Tr}(w)\right)\mathbf{a}^t - \sum_{u\in\mathbb{F}} \overline{\chi}(u^2)\psi(u)\mathbf{a}^t \\ &= \left(G_{p^2}\left(\overline{\chi}\circ N, \psi\circ\frac{1}{2}\mathrm{Tr}\right) - G_p\left(\overline{\chi}^2,\psi\right)\right)\mathbf{a}^t.\end{aligned}$$

From Proposition 1 (ii), the difference of the two Gauss sums in the last equation is not zero, and the result follows. □

Remark 1. Let $X(\delta,a)$ be a finite upper half plane graph, which is a Ramanujan graph when $a\neq 0, 4\delta$, and let A_a be the adjacency operator of $X(\delta,a)$. Then, it is easily seen that for any $a\in\mathbb{F}_p$, $F(z;\chi,\psi,\mathbf{a})$ is an eigenfunction of A_a.

3.2.2 Case $\boldsymbol{\Gamma = K_q}$ We use the notations in Section 2. For the multiplier system $m: K_q\times H_q\to\mathbb{C}^\times$ in (3), a complex representation $\rho: K_q\to GL(n,\mathbb{C})$, and a vector $\mathbf{b}\in\mathbb{C}^n$, let

$$E(z;\pi,\rho,\mathbf{b}) = \frac{1}{|K_q|}\sum_{k\in K_q}\frac{J_\pi(k,\sqrt{\delta})}{J_\pi(k,z)}\cdot\rho(k)^{-1}\mathbf{b}^t.$$

$E(z;\pi,\rho,\mathbf{b})$ is called the *vector-valued Eisenstein sum* for K_q, ρ, and $\mathbf{b}$. This sum, which is a finite field analog of the vector-valued Eisenstein series on H_q, is a generalization of the sum in (4).

Theorem 6. $E(z;\pi,\rho,\mathbf{b})\in M(K_q,m,\rho)$.

Proof. By (2), for $k' \in K_q$, it holds that

$$|K_q| E(k'z; \pi, \rho, \mathbf{b}) = \sum_{k \in K_q} \frac{J_\pi(kk', \sqrt{\delta})}{J_\pi(k', \sqrt{\delta})} \cdot \frac{J_\pi(k', z)}{J_\pi(kk', z)} \cdot \rho(k)^{-1} \mathbf{b}^t$$

$$= m(k', z)\rho(k') \sum_{k \in K_q} \frac{J_\pi(kk', \sqrt{\delta})}{J_\pi(kk', z)} \cdot \rho(kk')^{-1} \mathbf{b}^t,$$

and the result follows. □

4 Equivariant functions

4.1 Definitions

For a subgroup Γ of $G_q = GL(2, \mathbb{F}_q)$, let $\rho : \Gamma \to GL(2, \mathbb{C})$ be a 2-dimensional complex representation. The quotient $h(z) = f_1(z)/f_2(z)$ with $f_1, f_2 \in \mathcal{F}_q$ $(f_2(z) \neq 0)$ is called a *ρ-equivariant function* with respect to Γ if

$$h(\gamma z) = \rho(\gamma) \cdot h(z)$$

holds for $\gamma \in \Gamma$ and $z \in H_q$. Here the action on both sides is given by linear fractional transformations.

If $(f_1(z), f_2(z))^t \in M(\Gamma, m, \rho)$ with $f_2(z) \neq 0$, then it is easily seen that the quotient $f_1(z)/f_2(z)$ is a ρ-equivariant function. The following question is now raised: *for a given representation ρ, does there exist a ρ-equivariant function?* In the classical case, the corresponding problem was solved in [13, 14].

Let $n = 2$ in Example 2. If the lowest component of $E_\Gamma(z; \chi, \rho, \mathbf{a})$ is non-zero, then using this function, a ρ-equivariant function can be constructed.

4.2 Special cases

Let p be an odd prime number. Herein, ρ-equivariant functions are discussed when Γ is N_p or K_p.

4.2.1 Case $\boldsymbol{\Gamma = N_p}$ Using a generator t of $\mathbb{F}_p^\times$, a multiplicative character $\chi : \mathbb{F}_p^\times \to \mathbb{C}^\times$ is defined by $\chi(t) = e^{2\pi i/p}$. Moreover, an additive character $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ is defined by $\psi(1) = e^{2\pi i/p}$. Then, we have the following.

Theorem 7. *For any complex representation $\rho : N_p \to GL(2, \mathbb{C})$, there exists a ρ-equivariant function.*

Proof. By Theorem 1 (ii), the function $f(z; \chi, \psi)$ in (1) is non-zero. As N_p is abelian, ρ is equivalent to the direct sum of 1-dimensional representations $\alpha, \beta : N_p \to \mathbb{C}^\times$. When

$$\rho(\gamma_u) = \begin{pmatrix} \alpha(\gamma_u) & 0 \\ 0 & \beta(\gamma_u) \end{pmatrix} \qquad (\gamma_u \in N_p),$$

there exists i $(0 \leq i \leq p-1)$ such that $\alpha = \beta\mu^i$. Then, the pair of functions $(h_1(z), h_2(z))^t \in \mathcal{F}_q^2$ is defined as

$$(h_1(z), h_2(z))^t = (f(z;\chi,\psi)^{i+1}, f(z;\chi,\psi))^t \quad if\ \alpha = \beta\mu^i \quad (i = 0, 1, \ldots, p-1).$$

Then, for $\gamma_u \in N_p$, $h_1(\gamma_u z)/h_2(\gamma_u z) = \rho(\gamma_u) \cdot h_1(z)/h_2(z)$.

When there exists a matrix $U \in GL(2, \mathbb{C})$ such that

$$\rho(\gamma_u) = U \begin{pmatrix} \alpha(\gamma_u) & 0 \\ 0 & \beta(\gamma_u) \end{pmatrix} U^{-1} \qquad (\gamma_u \in N_p),$$

let $(f_1(z), f_2(z))^t = U(h_1(z), h_2(z))^t$. Then, for $\gamma_u \in N_p$, $f_1(\gamma_u z)/f_2(\gamma_u z) = \rho(\gamma_u) \cdot f_1(z)/f_2(z)$. □

4.2.2 Case $\boldsymbol{\Gamma = K_p}$

When $p = 3$, we have the following result.

Proposition 2. *Let $\rho : K_3 \to GL(2, \mathbb{C})$ be a complex representation such that $\rho(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, there exists a ρ-equivariant function.*

Proof. We use the notations in Example 1. By direct computation, $E(\sqrt{\delta}; \pi^4, 1) = 1$, $E(1+\sqrt{\delta}; \pi^4, \mu) = (1+i)/2$. Hence, we have non-zero modular forms $E(z; \pi^4, 1) \in M(K_3, m_4, 1)$ and $E(z; \pi^4, \mu) \in M(K_3, m_4, \mu)$. As K_3 is abelian, ρ is equivalent to the direct sum of 1-dimensional representations $\alpha, \beta : K_3 \to \mathbb{C}^\times$.

When

$$\rho(k) = \begin{pmatrix} \alpha(k) & 0 \\ 0 & \beta(k) \end{pmatrix} \qquad (k \in K_3),$$

by assumption, there exists j $(0 \leq j \leq 3)$ such that $\alpha = \beta\mu^j$. Then, the pair of functions $(h_1(z), h_2(z))^t \in \mathcal{F}_q^2$ is defined as

$$(h_1(z), h_2(z))^t = \begin{cases} (E(z;\pi^4, 1), E(z;\pi^4, 1))^t & if\ \alpha = \beta, \\ (E(z;\pi^4, \mu)^j, E(z;\pi^4, 1)^j)^t & if\ \alpha = \beta\mu^j\ (j = 1, 2, 3). \end{cases}$$

Then, for $k \in K_3$, $h_1(kz)/h_2(kz) = \rho(k) \cdot h_1(z)/h_2(z)$.

When there exists a matrix $U \in GL(2, \mathbb{C})$ such that

$$\rho(k) = U \begin{pmatrix} \alpha(k) & 0 \\ 0 & \beta(k) \end{pmatrix} U^{-1} \qquad (k \in K_3),$$

let $(f_1(z), f_2(z))^t = U(h_1(z), h_2(z))^t$. Then, for $k \in K_3$, $f_1(kz)/f_2(kz) = \rho(k) \cdot f_1(z)/f_2(z)$. □

5 Concluding remarks

Vector-valued modular forms on finite upper half planes have been introduced and then applied to equivariant functions. It is interesting to consider the following questions:

• It is difficult to determine the dimension of the space of vector-valued modular forms. Can a formula for its dimension be established?

• There is a shortage of interesting examples of the modular forms discussed in this paper. Can Poincaré series be defined on finite upper half planes?

• In [9], Hilbert modular forms on $H_q \times \cdots \times H_q$, i.e., the product of H_q, were introduced. Can Siegel modular forms be defined on "finite Siegel upper half spaces"? This may possibly be accomplished by starting with the finite symplectic group factored out by a stabilizer as in the case of H_q in Section 2. It is known that classical vector-valued modular forms are related to Jacobi forms [6]. Can an analog of Jacobi forms be defined?

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References

1. Angel, J., Celinker, N., Poulos, S., Terras, A., Trimble, C., Velasquez, E.: Special functions on finite upper half planes. Contemporary Math. **138**, 1–26 (1992)
2. Bantay, P.: Vector-valued modular forms. Contemporary Math. **497**, 19–31 (2009)
3. Celinker, N., Poulos, S., Terras, A., Trimble, C., Velasquez, E.: Is there life on finite upper half planes? Contemporary Math. **143**, 65–88 (1993)
4. Cohen, H., Strömberg, F.: Modular Forms. AMS, (2017)
5. Dong, C., Li, H., Mason, G.: Modular-invariance of trace functions in orbifold theory and generalized moonshine. Comm. Math. Phys. **214**, 1–56 (2000)
6. Eichler, M., Zagier, D.: The Theory of Jacobi Forms. Birkhäuser, (1985)
7. Eholzer, W., Skoruppa, N.-P.: Modular invariance and uniqueness of conformal characters. Comm. Math. Phys. **174**, 117–136 (1995)
8. Freitag, E., Busam, R.: Complex Analysis (Second Edition). Springer, (2009)
9. Hamahata, Y.: A note on modular forms on finite upper half planes. C. Carlet and B.Sunar (Eds.): WAIFI 2007, LNCS **4547**, pp. 18–24 (2007)
10. Knopp, M., Mason, G.: Generalized modular forms. J. Number Theory **99**, 1–28 (2003)
11. Knopp, M., Mason, G.: On vector-valued modular forms and their Fourier coefficients. Acta Arithmetica **110**, 117–124 (2003)
12. Knopp, M., Mason, G.: Vector-valued modular forms and Poincaré series. Illinois J. Math. **48**, 1345–1366 (2004)
13. Saber, H., Sebbar, A.: Equivariant functions and integrals of elliptic functions. Geom. Dedicata **160**, 373–414 (2012)
14. Saber, H, Sebbar, A.: Equivariant functions and vector-valued modular forms. International J. Number Theory **10**, 949–954 (2014)

15. Saber, H., Sebbar, A.: On the existence of vector-valued automorphic forms. Kyushu J. Math. **71**, 271–285 (2017)
16. Selberg, A.: On the estimation of Fourier coefficients of modular forms. Proc. Symp. Pure Math. VIII, AMS, 1–15 (1965)
17. Shaheen, A., Terras, A.: Fourier expansions of complex-valued Eisenstein series on finite upper half planes. Int. J. Math. Sci. **2006**, 1–17 (2006)
18. Terras, A.: Fourier Analysis on Finite Groups and Applications. London Mathematical Society Student Texts, vol. 43. Cambridge Univ. Press, Cambridge, (1999)
19. Terras, A.: Harmonic Analysis on Symmetric Spaces – Euclidean Space, the Sphere, and the Poincaré Upper Half-Plane (Second Edition). Springer, (2013)